Schubert Polynomial $\rightleftarrows$ div. differece / pipe dreams

How about Schur?

$\lambda = (\lambda_1, \dots, \lambda_k) \subset k \times (n-k)$ rectangle

$\rightsquigarrow w(\lambda) \in S_n$ Grassmanian permutation

Then $s_\lambda(x_1, \dots, x_k) = \mathfrak{S}_{w(\lambda)}$

Thm $s_\lambda(x_1, \dots, x_k) = \sum_{T \text{ is } SSYT(\lambda)} x^T \leftarrow (\#1, \#2, \dots)$

we will see that this is special case of pipe dream formula

Claim: pipe dreams P for $w(\lambda)$ are in bijection with $SSYT(\lambda)$.

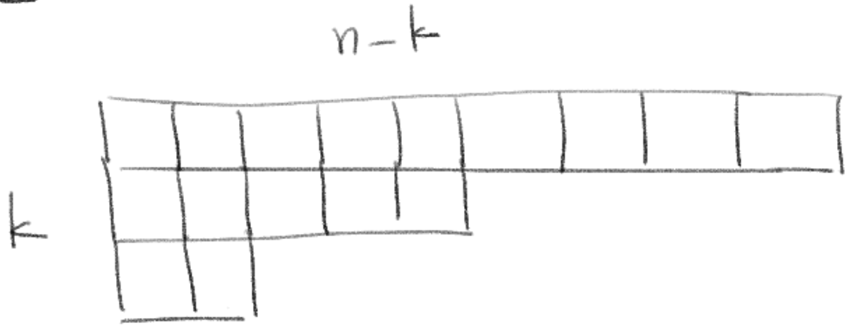
Rule: order the wires of P from top to bottom by their left end. For $i=1,\dots,k$

$\#$ crossings of i^{th} wire of P in j^{th} row

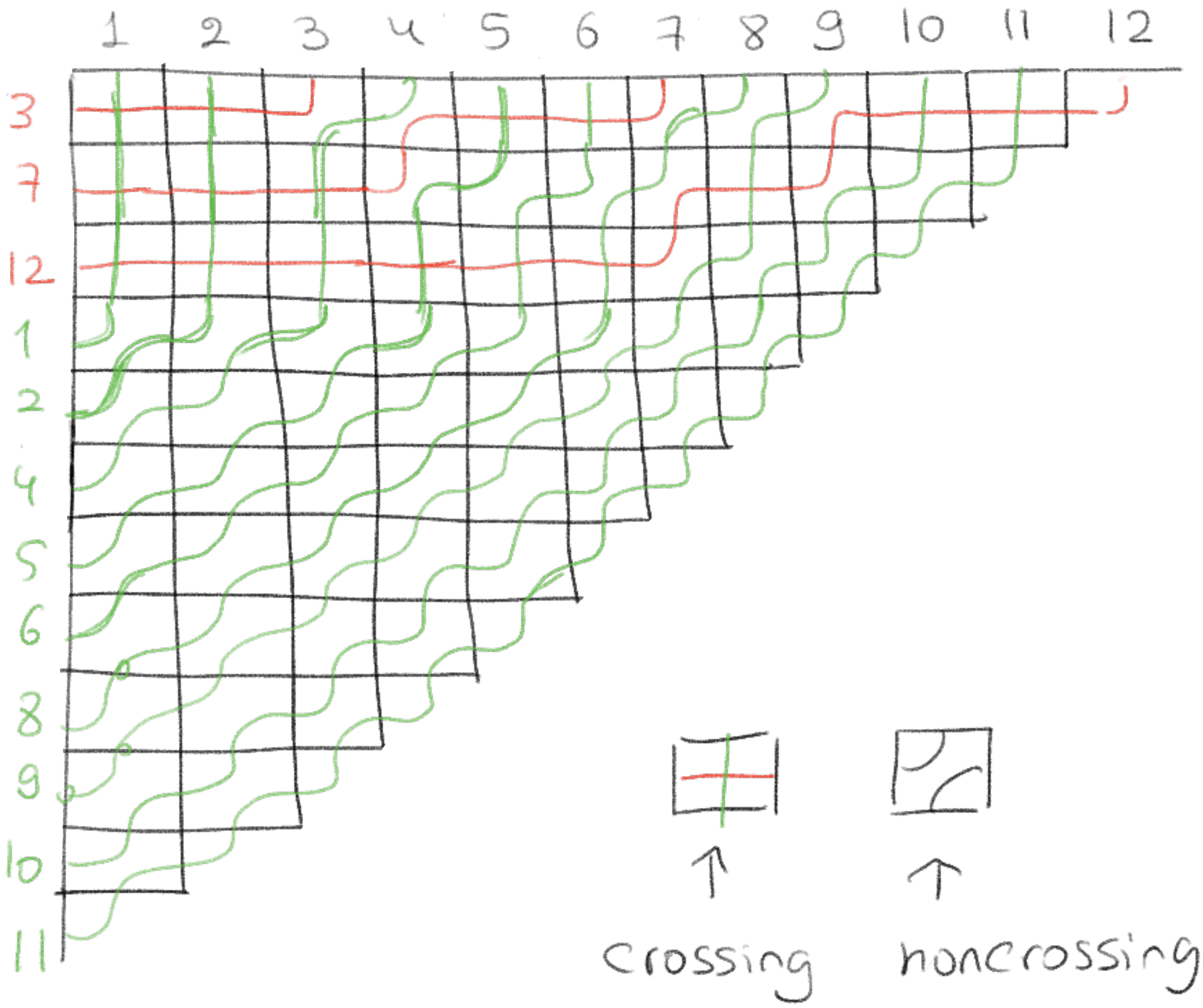
$= \#$ entries $k+1-j$ in $(k+1-i)^{th}$ row of T

Ex $n=12$ and $k=3$

$\lambda=(9,5,2)$



$$w(\lambda) = 2+1 \quad 5+2 \quad 9+3 \quad 1 \quad 2 \quad 4 \quad 5 \quad 6 \quad 8 \quad 9 \quad 10 \quad 11$$
$$= (3 \quad 7 \quad 12 \quad 1 \quad 2 \quad 4 \quad 5 \quad 6 \quad 8 \quad 9 \quad 10 \quad 11)$$



$T =$

1	1	1	1	1	1	2	3	3
2	2	2	3	3				
3	3							

Double Schubert Polynomials

$\mathfrak{S}_w(x,y) := \mathfrak{S}_w(x_1,\dots,x_{n-1},y_1,\dots,y_{n-1})$ for $w \in S_n$

defined as follows:

(1) $\mathfrak{S}_{w_0}(x,y) = \prod_{\substack{(i,j)\\ i+j\leq n}} (x_i - y_j)$

(2) $\mathfrak{S}_w(x,y) = \partial_i^x \mathfrak{S}_{ws_i}(x,y)$

if $\ell(ws_i) = \ell(w)+1$

where ∂_i^x only acts on x coordinates.

<u>claim</u> $\mathfrak{S}_w(x_1,\dots,x_{n-1},-y_1,-y_2,\dots,-y_{n-1})$

has positive integer coefficients.

Thm (Pipe dream formula for $\mathfrak{S}_w(x,y)$)

$$\mathfrak{S}_w(x,y) = \sum_{\substack{p \text{ is pipe} \\ \text{dream of } w}} \prod_{\substack{(i,j) \\ p \text{ has a crossing} \\ \text{in } i^{th} \text{ row and } j^{th} \text{ column}}} (x_i - y_j)$$

Cor (Symmetry) $\mathfrak{S}_w(x,y) = \mathfrak{S}_{w^{-1}}(-y,-x)$

For proof we work in Algebra over $\mathbb{C}$.

gens: $u_1,\dots,u_{n-1}$ $x_1,\dots,x_{n-1}$ $y_1,\dots,y_{n-1}$

rels: x_i, y_j commute with u_i-s and each other

(1) $u_i^2 = 0$

(2) $u_i u_j = u_j u_i$ $|i-j| \geq 2$

(3) $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$

$\rightarrow$ $h_i(a) = 1 + a u_i$ satisfy YB relations

$$S^{x,y} = \prod_{\substack{(i,j) \\ i+j\leq n}} h_{i+j-1}(x_i - y_j)$$

order (i,j) as follows

$$\begin{matrix} (1,n-1) & (1,n-2) & \cdots & & (1,1) \\ (2,n-2) & (2,n-2) & \cdots & (2,1) & \\ \vdots & & & & \\ (n-1,1) & & & & \end{matrix}$$

<u>Example</u> $(n=4)$

$h_1(x_1-y_1)$	$h_2(x_1-x_2)$	$h_3(x_1-y_3)$
$h_2(x_2-y_1)$	$h_3(x_2-y_2)$	
$h_3(x_3-y_1)$		

<u>Thm</u> $S^{x,y} = \sum_{w\in S_n} \mathfrak{S}_w(x,y)\, u_w$

It follows from

<u>**Lemma**</u> $\partial_i^x(S^{x,y}) = S^{x,y}u_i$